# Inertia Theorems for the Periodic Lyapunov Difference Equation and Periodic Riccati Difference Equation

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ABSTRACT

The periodic Lyapunov difference equation (PLDE) and periodic Riccati difference equation (PRDE) are dealt with. The inertia (i.e., the number of positive, null, and negative eigenvalues) of any symmetric periodic solution of such equations is linked with the pattern of eigenvalues of the monodromy matrix associated with the open-loop (for PLDE) or closed-loop (for PRDE) underlying systems. Different results are obtained by imposing requirements with decreasing strength to the original system. Precisely, assumptions of observability, reconstructibility, and detectability are successively introduced. Some results are also given for the particular case of positive semidefinite periodic solutions.

## 1. INTRODUCTION

The analysis of the Lyapunov and Riccati equations is widely recognized to be fundamental in many topics of linear systems theory. In particular, the Lyapunov equation is a basic tool in the study of linear systems stability, whereas the Riccati equation arises in many problems related to optimal filtering and control. Depending on whether continuous-time or discrete-time systems are considered, one has to handle either differential or difference equations, respectively.

Some authors have investigated the properties of the solutions of those equations in connection with the structural properties of the underlying




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system. Such results fall under the heading of "inertia theorems." Precisely, in the time-invariant case, it has been shown that, under the assumption of system observability, the eigenvalues of the solutions are linked in a direct way with the eigenvalues of the system dynamic matrix (see, e.g., [17, 11, 19, 21, 20]). These results have proved to be useful in the stability analysis of multivariable linear control systems [13, 14] and in the development of the theory of the Riccati equation [12, 16]. Recently, the more general case of continuous-time periodic systems has been considered in [18]. The main results of that paper can be seen as an extension to the periodic case of the theorems derived in [21, 20]. As a matter of fact, it is proved in [18] that if the system is observable, a close relationship exists between the eigenvalues of the periodic solutions of the periodic Lyapunov and Riccati differential equations and the eigenvalues of the so-called system monodromy matrix. By weakening the assumption of observability to detectability, some new inertia theorems for the continuous-time periodic case have been obtained in [6] and [7].

The aim of the present paper is to provide inertia theorems for the periodic Lyapunov and Riccati difference equations. As far as the assumptions of observability and detectability are concerned, the results presented in this paper turn out to be an extension to the discrete-time context of the theorems given in [18, 6, 7]. However, when switching from continuous time to discrete time, a more intriguing and varied problem is to be tackled. Actually, it is well known (see e.g. [15]) that, contrary to the continuous-time case, the properties of observability and reconstructibility for a nonreversible discrete-time linear periodic system are not equivalent to each other. Precisely, system reconstructibility (state determination using past outputs) is only a necessary condition for system observability (state determination using future outputs). As a consequence, starting from the weaker assumption of reconstructibility instead of observability, new additional inertia theorems can be formulated.

All the proofs in this paper basically rely on modal characterizations of observability, reconstructibility, and detectability for discrete-time periodic systems. Such characterizations are discussed in detail in [3, 9]. Actually, those papers deal with reachability, controllability, and stabilizability, which are the dual notions of observability, reconstructibility, and detectability, respectively. In this respect, it is worthwhile noticing that all the results contained in this paper can be easily extended by duality to the case where the dual versions of the Lyapunov and Riccati equations are considered.

The paper is organized as follows. In Section 2, the concepts and the notation that will be used throughout the paper are introduced. In particular, the modal characterizations of observability, reconstructibility, and detectability are presented here and briefly discussed. The inertia theorems for the periodic Lyapunov difference equation and the periodic Riccati difference equation are derived in Sections 3 and 4, respectively. Some final comments are reported in Section 5.

## 2. PRELIMINARIES

The class of systems dealt with in this paper is described by

$$x(t+1) = A(t)x(t) + B(t)u(t), \tag{1.a}$$

$$y(t) = C(t)x(t), \tag{1.b}$$

where $t \in Z$ (the set of integers), $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$, and $A(\cdot), B(\cdot), C(\cdot)$ are real $T$-periodic matrix functions of suitable dimensions, i.e.,

$$A(t+T) = A(t), \quad B(t+T) = B(t), \quad C(t+T) = C(t) \qquad \forall t.$$

Let $\Phi(t,\tau)$, $t \geqslant \tau$, be the transition matrix of the system (1) (see the Appendix for a precise definition). Then $\overline{\Phi}_t = \Phi(t+T, t)$ is usually called the monodromy matrix at time $t$. Its eigenvalues are independent of $t$. Moreover, the system (1) is asymptotically stable if and only if all the eigenvalues of $\overline{\Phi}_t$ lie inside the open unit disk.

It is also useful to define the observability Gramian matrix of system (1), given by

$$W(\tau, t) = \sum_{j=\tau}^{t-1} \Phi(j,\tau)'C(j)'C(j)\Phi(j,\tau), \qquad \tau < t.$$

In the sequel, we will often refer to the observability Gramian over one period, i.e.,

$$\overline{W}_t = W(t, t+T).$$

It is well known (see the Appendix) that the null space of $W(\tau, t)$ corresponds to the unobservability subspace of the system (1) over the interval $[\tau, t]$. Moreover, it can be shown (see, e.g., [2]) that the unobservability subspace at time $t$ ($\overline{X}_t^{\omega}$) is given by the null space of $W(t, t+nT)$, i.e.,

$$\overline{X}_t^{\omega} = N[W(t, t+nT)].$$

The system (1) is said to be observable at time $t$ if $\bar{X}_t^{\omega} = \{0\}$. Besides this classical notion of observability, a different modal characterization can also be introduced, which is particularly suitable for our discussion. This characterization of observability, together with the analogous ones for reconstructibility and detectability, is reported below for ease of reference. The equivalence between these definitions and the classical ones has been proved in [3, 9]. Parallel results valid for continuous-time periodic systems can be found in [8, 4].

OBSERVABILITY. The system (1) [or the pair $(A(\cdot), C(\cdot))$] is observable at time $t$ if and only if, for each eigenvalue $\lambda$ of $\bar{\Phi}_t$, $\bar{\Phi}_t x = \lambda x$ and $C(j)\Phi(j, t)x = 0$, $j \in [t, t+T-1]$, imply $x = 0$.

RECONSTRUCTIBILITY. The system (1) [or the pair $(A(\cdot), C(\cdot))$] is reconstructible at time $t$ if and only if, for each eigenvalue $\lambda \neq 0$ of $\bar{\Phi}_t$, $\bar{\Phi}_t x = \lambda x$ and $C(j)\Phi(j, t)x = 0$, $j \in [t, t+T-1]$, imply $x = 0$.

DETECTABILITY. The system (1) [or the pair $(A(\cdot), C(\cdot))$] is detectable at time $t$ if and only if, for each eigenvalue $\lambda$ of $\bar{\Phi}_t$ with $|\lambda| \geqslant 1$, $\bar{\Phi}_t x = \lambda x$ and $C(j)\Phi(j, t)x = 0$, $j \in [t, t+T-1]$, imply $x = 0$.

In [3] it is proved that if the system (1) is reconstructible at an arbitrary time point $\bar{t}$, it is reconstructible at any time point. The same is true for detectability (see [9]). This is not surprising, since detectability can be equivalently defined in terms of stability of the unreconstructible part of the system, and thus it is a property independent of $t$. On the contrary, system observability at a given $\bar{t}$ does not imply observability at a different time point unless the system is reversible (again see [3]). Moreover, it is apparent from the above definitions that observability at a fixed $\bar{t}$ implies reconstructibility at any $t$, which, in turn, implies detectability at any $t$.

Notice that, when specialized to a time-invariant system of the form

$$x(t+1) = Fx(t), \tag{2.a}$$

$$y(t) = Hx(t), \tag{2.b}$$

$x(t) \in R^n$, $y(t) \in R^p$, the three definitions above turn out to coincide with the usual PBH-like characterizations of observability, reconstructibility, and detectability (see, e.g., [15]). Precisely, the system (2) [or the pair $(F, H)$] is observable if and only if, for each eigenvalue $\lambda$ of $F$, $Fx = \lambda x$ and $Cx = 0$ imply $x = 0$. The analogous definitions of reconstructibility and detectability are simply obtained by considering only the eigenvalues of $F$ with $\lambda \neq 0$ and $|\lambda| \geqslant 1$, respectively.

Associated with the system (1), consider the periodic Lyapunov difference equation (PLDE)

$$P(t) = A(t)'P(t+1)A(t) + C(t)'C(t)$$

and the periodic Riccati difference equation (PRDE)

$$P(t) = A(t)'P(t+1)A(t) + C(t)'C(t)$$
$$- A(t)'P(t+1)B(t)[I + B(t)'P(t+1)B(t)]^{\dagger}B(t)'P(t+1)A(t),$$

where $'$ denotes transpose and $^{\dagger}$ denotes the Moore-Penrose inverse operation. The purpose of the present work is to analyze the properties of the symmetric $T$-periodic solutions of the PLDE and PRDE.

To conclude this section, let us introduce the following short notation to denote the (continuous or discrete) spectrum of a square matrix S. The symbols $\nu_c(S)$, $\delta_c(S)$, and $\pi_c(S)$ will represent the numbers of eigenvalues of $S$ with negative, zero, and positive real part, respectively. The symbols $\nu_d(S)$, $\delta_d(S)$, and $\pi_d(S)$ will represent the numbers of eigenvalues of S with modulus less than, equal to, and greater than 1, respectively.

## 3. INERTIA THEOREMS FOR THE PLDE

In this section we will provide some inertia theorems for the periodic Lyapunov difference equation (PLDE). Under suitable assumptions on the pair $(A(\cdot), C(\cdot))$ (observability, reconstructibility, detectability), these theorems point out the relationship between the numbers of positive, zero, and negative eigenvalues of any symmetric periodic solution of the PLDE at a given time $t$ and the numbers of eigenvalues of the monodromy matrix that lie inside, on, and outside the unit circle in the complex plane.

In the following derivation, a key role will be played by the so-called discrete algebraic Lyapunov equation (DALE)

$$Q = F'QF + H'H$$

associated with the system (2). For the DALE a number of inertia theorems are available in the literature. They can be summarized as follows.

LEMMA 1. *Suppose that a symmetric solution $\overline{Q}$ of the* DALE *exists.*

(i) *If the pair $(F, H)$ is observable, then*

$$\pi_c(\overline{Q}) = \nu_d(F), \qquad \nu_c(\overline{Q}) = \pi_d(F), \qquad \delta_c(\overline{Q}) = \delta_d(F) = 0.$$

(ii) *If the pair* $(F, H)$ *is reconstructible, then*

$$\pi_c(\overline{Q}) = \nu_d(F) - \bar{q}^{\omega}, \qquad \nu_c(\overline{Q}) = \pi_d(F), \qquad \delta_c(\overline{Q}) = \bar{q}^{\omega}, \qquad \delta_d(F) = 0,$$

*where* $\bar{q}^{\omega}$ *is the dimension of the unobservability subspace of the system* (2).

(iii) *If the pair* $(F, H)$ *is detectable, then*

$$\pi_c(\overline{Q}) + \delta_c(\overline{Q}) = \nu_d(F), \qquad \nu_c(\overline{Q}) = \pi_d(F), \qquad \delta_d(F) = 0.$$

(iv) *If the pair* $(F, H)$ *is detectable and* $\overline{Q}$ *is the unique solution of the* DALE, *then the same result as in* (ii) *holds.*

The proofs can be found in [21] [point (i)], [10] [points (ii) and (iv)], and [6] [point (iii)].

In order to derive the extension of this lemma to the PLDE, two more preliminary results are needed. They are given in Lemmas 2 and 3, below.

LEMMA 2. *If* $\overline{P}(\cdot)$ *is a symmetric T-periodic solution of the* PLDE, *then for any* $t$, $\overline{P}_t = \overline{P}(t)$ *is a symmetric solution of the following* DALE:

$$\overline{P}_t = \overline{\Phi}_t' \overline{P}_t \overline{\Phi}_t + \overline{D}_t' \overline{D}_t, \tag{3}$$

*where* $\overline{D}_t$ *is an arbitrary matrix such that* $\overline{D}_t' \overline{D}_t = \overline{W}_t$.

*Proof.* It is easy to show that the backward solution of the PLDE, starting from $\overline{P}(t + T) = \overline{P}_t$ is given by

$$\overline{P}(\tau) = \Phi(t + T, \tau)' \overline{P}_t \Phi(t + T, \tau) + W(\tau, t + T), \qquad \tau < t + T.$$

By imposing periodicity, i.e., $\overline{P}(t) = \overline{P}_t$ the conclusion directly follows. ■

LEMMA 3. *Consider the pair* $(A(\cdot), C(\cdot))$ *and the corresponding pair* $(\overline{\Phi}_t, \overline{D}_t)$.

(i) *If* $(A(\cdot), C(\cdot))$ *is observable at time* $t$, *then* $(\overline{\Phi}_t, \overline{D}_t)$ *is observable.*

(ii) *If* $(A(\cdot), C(\cdot))$ *is reconstructible at time* $t$, *then* $(\overline{\Phi}_t, \overline{D}_t)$ *is reconstructible.*

(iii) *If* $(A(\cdot), C(\cdot))$ *is detectable at time* $t$, *then* $(\overline{\Phi}_t, \overline{D}_t)$ *is detectable.*

*Proof.* (i): Suppose by contradiction that the pair $(\overline{\Phi}_t, \overline{D}_t)$ is not observable. Then there exist an eigenvalue $\lambda$ of $\overline{\Phi}_t$ and a vector $x \neq 0$ such that

$$\overline{\Phi}_t x = \lambda x, \tag{4}$$

$$\overline{D}_t x = 0. \tag{5}$$

Using * to denote conjugate transpose, it follows from (5) that

$$x^*\overline{D}_t'\overline{D}_t x = x^*\overline{W}_t x = \sum_{j=t}^{t+T-1} x^*\Phi(j,t)'C(j)'C(j)\Phi(j,t)x = 0.$$

Hence, it is apparent that

$$C(j)\Phi(j,t)x = 0, \qquad j \in [t, t+T-1]. \tag{6}$$

Finally, (4), (6), and $x \neq 0$ lead to the contradiction.

(ii) (iii): The proofs are completely analogous except for the fact that only $\lambda \neq 0$ and $|\lambda| \geqslant 1$, respectively, are considered. ∎

Notice that the sufficient conditions stated in Lemma 3 are in fact also necessary. However, the point is quite irrelevant to our purpose.

We are now in a position to prove the first inertia theorem for the PLDE.

THEOREM 1. *Let the* PLDE *admit a symmetric T-periodic solution* $\overline{P}(\cdot)$, *and suppose that* $(A(\cdot), C(\cdot))$ *is observable at time* $t$. *Then*

$$\pi_c(\overline{P}(t)) = \nu_d(\overline{\Phi}_0), \tag{7.a}$$

$$\nu_c(\overline{P}(t)) = \pi_d(\overline{\Phi}_0), \tag{7.b}$$

$$\delta_c(\overline{P}(t)) = \delta_d(\overline{\Phi}_0) = 0. \tag{7.c}$$

*Proof.* Lemma 2 implies that $\overline{P}_t = \overline{P}(t)$ is a solution of the DALE (3). Moreover, in view of Lemma 3, the pair $(\overline{\Phi}_t, \overline{D}_t)$ is observable. Hence, by applying Lemma 1(i) to (3) and recalling that the eigenvalues of $\overline{\Phi}_t$ are independent of $t$, the conclusion follows. ∎

In view of Theorem 1 it is easy to conclude that if the pair $(A(\cdot), C(\cdot))$ is observable at any $t$, the equations (7) hold for any $t$ as well.

We will now consider the case of reconstructibility instead of observability. To this aim, another relationship between the pairs $(A(\cdot), C(\cdot))$ and $(\overline{\Phi}_t, \overline{D}_t)$ has to be first pointed out.

LEMMA 4. *The unobservability subspace of the pair* $(A(\cdot), C(\cdot))$ *at time* $t$ *coincides with the unobservability subspace of the pair* $(\overline{\Phi}_t, \overline{D}_t)$.

*Proof.* Let $\overline{X}_t^{\omega}$ denote the unobservability subspace associated with the pair $(A(\cdot), C(\cdot))$ at time $t$, and let $\overline{Z}_t^{\omega}$ denote the unobservability subspace associated with the pair $(\overline{\Phi}_t, \overline{D}_t)$. As previously recalled in Section 2, $\overline{X}_t^\omega$ can be expressed as

$$\overline{X}_t^\omega = N\left[W(t, t+nT)\right].$$

By considering the definition of the Gramian and applying periodicity one can obtain

$$\begin{aligned}
W(t, t+nT) &= \sum_{j=t}^{t+nT-1} \Phi(j,t)'C(j)'C(j)\Phi(j,t) \\
&= \sum_{k=0}^{n-1} \sum_{i=t+kT}^{t+(k+1)T-1} \Phi(i,t)'C(i)'C(i)\Phi(i,t) \\
&= \sum_{k=0}^{n-1} \Phi(t+kT,t)'\left[\sum_{i=t+kT}^{t+(k+1)T-1} \Phi(i,t+kT)'C(i)'C(i)\Phi(i,t+kT)\right] \\
&\quad \times \Phi(t+kT,t) \\
&= \sum_{k=0}^{n-1} \Phi(t+kT,t)' \\
&\quad \times \left[\sum_{l=t}^{t+T-1} \Phi(l+kT,t+kT)'C(l+kT)'C(l+kT)\Phi(l+kT,t+kT)\right] \\
&\quad \times \Phi(t+kT,t) \\
&= \sum_{k=0}^{n-1} (\overline{\Phi}_t')^k\left[\sum_{l=t}^{t+T-1} \Phi(l,t)'C(l)'C(l)\Phi(l,t)\right](\overline{\Phi}_t)^k \\
&= \sum_{k=0}^{n-1} (\overline{\Phi}_t')^k \overline{W}_t (\overline{\Phi}_t)^k. \qquad (8)
\end{aligned}$$

On the other hand, the unobservability subspace for the constant pair $(\overline{\Phi}_t, \overline{D}_t)$ ($t$ is fixed in this discussion) is given by

$$\overline{Z}_t^\omega = N\left[O_t'\right],$$

where

$$O_t = \left[\overline{D}_t' \quad \overline{\Phi}_t'\overline{D}_t' \quad \cdots \quad (\overline{\Phi}_t')^{n-1}\overline{D}_t'\right].$$

Now, it is apparent from (8) that

$$W(t, t+nT) = O_t O_t'.$$

Since $N[O_t O_t'] = N[O_t']$, the conclusion is drawn that $\overline{X}_t^\omega = \overline{Z}_t^\omega$. ∎

It is worthwhile noticing that the dimension of the unobservability subspace $\overline{X}_t^\omega$ generally depends on $t$ (see, e.g., [2]). This is a peculiar difference between discrete-time and continuous-time periodic systems. The dimension of $\overline{X}_t^\omega$ will be denoted by $\overline{n}_t^\omega$.

THEOREM 2. *Let the* PLDE *admit a symmetric T-periodic solution* $\overline{P}(\cdot)$, *and suppose that* $(A(\cdot), C(\cdot))$ *is reconstructible. Then, for any* $t$,

$$\pi_c(\overline{P}(t)) = \nu_d(\overline{\Phi}_0) - \overline{n}_t^\omega, \tag{9.a}$$

$$\nu_c(\overline{P}(t)) = \pi_d(\overline{\Phi}_0), \tag{9.b}$$

$$\delta_c(\overline{P}(t)) = \overline{n}_t^\omega, \tag{9.c}$$

$$\delta_d(\overline{\Phi}_0) = 0. \tag{9.d}$$

*Proof.* In view of Lemmas 2, 3(ii), and (4), it is clear that

(i) $\overline{P}_t = \overline{P}(t)$ is a solution of the DALE (3);
(ii) the pair $(\overline{\Phi}_t, \overline{D}_t)$ is reconstructible;
(iii) the unobservability subspace of $(\overline{\Phi}_t, \overline{D}_t)$ has dimension equal to $\overline{n}_t^\omega$.

Then, since the eigenvalues of $\overline{\Phi}_t$ are independent of $t$, Lemma 1(ii) entails (9). ∎

By further weakening the assumption on the system (1), Theorems 3 and 4 below can be finally derived along the same line of reasoning used in the proofs of Theorems 1 and 2.

THEOREM 3. *Let the* PLDE *admit a symmetric T-periodic solution* $\overline{P}(\cdot)$, *and suppose that* $(A(\cdot), C(\cdot))$ *is detectable. Then, for any* $t$,

$$\pi_c(\overline{P}(t)) + \delta_c(\overline{P}(t)) = \nu_d(\overline{\Phi}_0), \tag{10.a}$$

$$\nu_c(\overline{P}(t)) = \pi_d(\overline{\Phi}_0), \tag{10.b}$$

$$\delta_d(\overline{\Phi}_0) = 0. \tag{10.c}$$

*Proof.* The result follows from Lemmas 2, 3(iii), and 1(iii). ∎

THEOREM 4. *Let the* PLDE *admit a* unique *symmetric T-periodic solution* $\bar{P}(\cdot)$, *and suppose that* $(A(\cdot), C(\cdot))$ *is detectable. Then, for any* $t$, *the relations* (9) *are verified.*

*Proof.* A moment's reflection reveals that $\bar{P}_t = \bar{P}(t)$ is the unique symmetric solution of (3). Hence, the result follows from Lemmas 2, 3(iii), and 1(iv). ■

When analyzing problems of stability for discrete-time periodic linear systems, one is usually interested in considering only positive semidefinite $T$-periodic solutions of the PLDE. In this respect, the results of this section can be specialized in the following corollary.

COROLLARY 1. *Let the* PLDE *admit a T-periodic solution* $\bar{P}(\cdot)$ *that is positive semidefinite at least at a particular* $t = \bar{t}$, *and suppose that* $(A(\cdot), C(\cdot))$ *is detectable. Then*

(i) *the system* (1) *is asymptotically stable*;
(ii) $\bar{P}(t)$ *is positive semidefinite for any* $t$;
(iii) $\bar{P}(\cdot)$ *is the unique T-periodic solution of the* PLDE;
(iv) *for any* $t$, *the number of zero eigenvalues of* $\bar{P}(t)$ *is equal to the dimension of the unobservability subspace of* $(A(\cdot), C(\cdot))$ *at time* $t$.

*Proof.* (i): Since $\nu_c(\bar{P}(\bar{t})) = 0$, from Theorem 3 it is apparent that $\bar{\Phi}_0$ has all eigenvalues inside the unit circle. Hence, the system (1) is asymptotically stable.

(ii): Directly from (10.b) it follows that $\nu_c(\bar{P}(t)) = 0\ \forall t$, i.e., $\bar{P}(t) \geqslant 0\ \forall t$.

(iii): This conclusion is a direct consequence of the asymptotic stability of the system (1) (see, e.g., [9]).

(iv): The solution being unique, Theorem 4 entails that $\delta_c(\bar{P}(t)) = \bar{n}_t^{\omega}\ \forall t$. ■

The result stated in Corollary 1 is closely related to the so-called Lyapunov lemma for discrete-time periodic systems. This topic is discussed in [1].

## 4. INERTIA THEOREMS FOR THE PRDE

A simple way to obtain inertia theorems for the periodic Riccati difference equation (PRDE) consists of recognizing that any solution of the PRDE is also a solution of a suitably defined PLDE. This enables one to apply the results presented in Section 3 to this latter equation. For instance, this approach has been followed in [18, 6, 7] in the continuous-time case.

LEMMA 5. *Let $\overline{P}(\cdot)$ be a symmetric T-periodic solution of the* PRDE. *Then $\overline{P}(\cdot)$ is a symmetric T-periodic solution of the following* PLDE:

$$P(t) = \hat{A}(t)'P(t+1)\hat{A}(t) + \hat{C}(t)'\hat{C}(t), \tag{11}$$

*where, for any $t$,*

$$\hat{A}(t) = A(t) - B(t)\left[I + B(t)'\overline{P}(t+1)B(t)\right]^{\dagger}B(t)'\overline{P}(t+1)A(t) \tag{12}$$

*and $\hat{C}(t)$ is such that*

$$\begin{aligned}\hat{C}(t)'\hat{C}(t) &= C(t)'C(t) + A(t)'\overline{P}(t+1)B(t)\\ &\quad\times\left\{\left[I + B(t)'\overline{P}(t+1)B(t)\right]^{\dagger}\right\}^{2}B(t)'\overline{P}(t+1)A(t).\end{aligned} \tag{13}$$

*Proof.* First, observe that, from (12), (13) and the periodicity of $\overline{P}(\cdot)$, the matrices $\hat{A}(\cdot)$ and $\hat{C}(\cdot)$ turn out to be $T$-periodic as well. Then the proof is easily carried out by substituting $\overline{P}(t)$ and the expressions (12) and (13) in (11). ∎

Consider now the pair $(\hat{A}(\cdot), \hat{C}(\cdot))$ defined by (12) and (13) in correspondence with an arbitrary symmetric $T$-periodic matrix function $\overline{P}(\cdot)$. In the following lemma it is shown that the properties of observability, reconstructibility, and detectability of the pair $(A(\cdot), C(\cdot))$ hold unchanged for the pair $(\hat{A}(\cdot), \hat{C}(\cdot))$. In the sequel, we will indicate by $\hat{\Phi}(t,\tau)$, $t \geqslant \tau$, the transition matrix associated with $\hat{A}(\cdot)$, and by $\hat{\overline{\Phi}}_t = \hat{\Phi}(t+T,t)$ the corresponding monodromy matrix at time $t$.

LEMMA 6. *Let $\overline{P}(\cdot)$ be a symmetric T-periodic function, and consider the pair $(\hat{A}(\cdot), \hat{C}(\cdot))$ defined by* (12) *and* (13).

(i) *If $(A(\cdot), C(\cdot))$ is observable at time $t$, then $(\hat{A}(\cdot), \hat{C}(\cdot))$ is observable at $t$.*

(ii) *If* $(A(\cdot), C(\cdot))$ *is reconstructible at time* $t$, *then* $(\hat{A}(\cdot), \hat{C}(\cdot))$ *is reconstructible at* $t$.

(iii) *If* $(A(\cdot), C(\cdot))$ *is detectable at time* $t$, *then* $(\hat{A}(\cdot), \hat{C}(\cdot))$ *is detectable at* $t$.

*Proof.* (i): By contradiction, suppose that $(\hat{A}(\cdot), \hat{C}(\cdot))$ is not observable at $t$. This means that there exist $\lambda$ and $x \neq 0$ such that

$$\hat{\Phi}(t+T, t)x = \lambda x, \tag{14}$$

$$\hat{C}(j)\hat{\Phi}(j, t)x = 0, \qquad j \in [t, t+T-1]. \tag{15}$$

By using (15), it is obvious that

$$x^*\hat{\Phi}(j,t)'\hat{C}(j)'\hat{C}(j)\hat{\Phi}(j,t)x = 0, \qquad j \in [t, t+T-1].$$

Then, in view of (13),

$$\begin{aligned} &x^*\hat{\Phi}(j,t)'C(j)'C(j)\hat{\Phi}(j,t)x + x^*\hat{\Phi}(j,t)'A(j)'\bar{P}(j+1)B(j) \\ &\quad \times \left\{ [I + B(j)'\bar{P}(j+1)B(j)]^{\dagger} \right\}^2 B(j)'\bar{P}(j+1)A(j)\hat{\Phi}(j,t)x \\ &\quad = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad j \in [t, t+T-1]. \end{aligned} \tag{16}$$

Since both terms on the left-hand side of (16) are nonnegative, they must be actually equal to zero separately. This, in turn, implies that

$$C(j)\hat{\Phi}(j,t)x = 0, \qquad j \in [t, t+T-1], \tag{17}$$

$$\begin{aligned} &[I + B(j)'\bar{P}(j+1)B(j)]^{\dagger}B(j)'\bar{P}(j+1)A(j)\hat{\Phi}(j,t)x = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad j \in [t, t+T-1]. \end{aligned} \tag{18}$$

By using (18), it results that

$$\begin{aligned} \hat{\Phi}(j+1,t)x &= \hat{A}(j)\hat{\Phi}(j,t)x \\ &= \left\{ A(j) - B(j)[I + B(j)'\bar{P}(j+1)B(j)]^{\dagger} \right. \\ &\qquad \left. \times B(j)'\bar{P}(j+1)A(j) \right\} \hat{\Phi}(j,t)x \\ &= A(j)\hat{\Phi}(j,t)x, \qquad j \in [t, t+T-1]. \end{aligned} \tag{19}$$

On the other hand, the very definition of the transition matrix yields

$$\Phi(j+1,t)x = A(j)\Phi(j,t)x \qquad \forall j \geq t. \tag{20}$$

Comparing (19) and (20) and noticing that $\hat{\Phi}(t,t)x = \Phi(t,t)x = x$, it should be apparent that

$$\hat{\Phi}(j,t)x = \Phi(j,t)x, \qquad j \in [t, t+T]. \tag{21}$$

Substituting (21) into (14) and (17), we finally obtain the contradicting conclusion that $(A(\cdot), C(\cdot))$ is not observable at $t$. Thus, the proof is completed.

(ii)–(iii): The proof of point (i) can be repeated by considering only eigenvalues $\lambda$ of $\hat{\Phi}(t+T,t)$ with $\lambda \neq 0$ and $|\lambda| \geq 1$, respectively. ■

Thanks to Lemmas 5 and 6, we have now all the ingredients to derive the main results of this section, which are stated in Theorems 5, 6, and 7, below. In fact, these theorems can be easily proved by applying Theorems 1, 2, and 3 of Section 3 to the particular PLDE (11) that is obtained from the PRDE once a solution $\bar{P}(\cdot)$ is known.

THEOREM 5. *Let the* PRDE *admit a symmetric T-periodic solution $\bar{P}(\cdot)$, and suppose that $(A(\cdot), C(\cdot))$ is observable at time $t$. Then*

$$\pi_c(\bar{P}(t)) = \nu_d(\hat{\bar{\Phi}}_0), \qquad \nu_c(\bar{P}(t)) = \pi_d(\hat{\bar{\Phi}}_0), \qquad \delta_c(\bar{P}(t)) = \delta_d(\hat{\bar{\Phi}}_0) = 0.$$

THEOREM 6. *Let the* PRDE *admit a symmetric T-periodic solution $\bar{P}(\cdot)$, and suppose that $(A(\cdot), C(\cdot))$ is reconstructible. Then, for any $t$,*

$$\pi_c(\bar{P}(t)) = \nu_d(\hat{\bar{\Phi}}_0) - \hat{n}_t^{\omega}, \qquad \nu_c(\bar{P}(t)) = \pi_d(\hat{\bar{\Phi}}_0),$$

$$\delta_c(\bar{P}(t)) = \hat{n}_t^{\omega}, \qquad \delta_d(\hat{\bar{\Phi}}) = 0,$$

*where $\hat{n}_t^{\omega}$ is the dimension of the unobservability subspace of the pair $(\hat{A}(\cdot), \hat{C}(\cdot))$ at time $t$.*

THEOREM 7. *Let the* PRDE *admit a symmetric T-periodic solution $\bar{P}(\cdot)$, and suppose that $(A(\cdot), C(\cdot))$ is detectable. Then, for any $t$,*

$$\pi_c(\bar{P}(t)) + \delta_c(\bar{P}(t)) = \nu_d(\hat{\bar{\Phi}}_0), \qquad \nu_c(\bar{P}(t)) = \pi_d(\hat{\bar{\Phi}}_0), \qquad \delta_d(\hat{\bar{\Phi}}_0) = 0.$$

Notice that the counterpart of Theorem 4 for the PRDE cannot be directly derived along the same line of reasoning, since there is no evidence that the uniqueness of the solution of a PRDE implies the uniqueness of the solution of the corresponding PLDE (11).

Observe now that, from the definition (12), $\hat{A}(\cdot)$ is the dynamic matrix of the closed-loop system obtained from the system (1) through the feedback control law

$$u(t) = -\left[I + B(t)'\bar{P}(t+1)B(t)\right]^{-1}B(t)'\bar{P}(t+1)A(t)x(t). \tag{22}$$

Hence, $\hat{\bar{\Phi}}_t$ can be interpreted as the monodromy matrix at time $t$ of such a system, and its eigenvalues determine the stability of the closed-loop system. Following this interpretation, a significant corollary can be derived from Theorem 7, above, the proof of which is completely analogous to that of Corollary 1 in Section 3.

COROLLARY 2. *Let the* PRDE *admit a symmetric T-periodic solution* $\bar{P}(\cdot)$ *that is positive semidefinite at least at a particular* $t = \bar{t}$, *and suppose that* $(A(\cdot), C(\cdot))$ *is detectable. Then*

(i) *the closed-loop system obtained from* (1) *through the feedback control law* (22) *is asymptotically stable;*

(ii) $\bar{P}(t)$ *is positive semidefinite for any* $t$.

As a final remark, observe that the inertia theorems presented in this section can be straightforwardly specialized to the time-invariant case, when the algebraic Riccati equation is considered.

## 5. CONCLUDING REMARKS

The inertia theorems presented in this paper are an extension to the periodic discrete-time case of some results recently obtained for the periodic Lyapunov differential equation and periodic Riccati differential equation. The theorems stated in terms of reconstructibility are peculiar to the discrete-time case, since the properties of observability and reconstructibility actually coincide for continuous-time periodic systems. This basic difference is essentially due to the fact that a discrete-time system may be nonreversible. Standard situations in which nonreversible periodic systems must be handled arise in the analysis of multirate sampled-data systems; see e.g. [22].

The results provided in this paper concern the properties of the periodic solutions of PLDE and PRDE under the assumption that these solutions exist.

In the development of a complete theory for PLDE and PRDE, further investigation has to be carried out in order to assess different results, such as the existence and uniqueness of the periodic solution or convergence of a solution to the periodic one. Research on these topics is currently underway.

## APPENDIX

In this appendix the basic definitions of observability, reconstructibility, and detectability for discrete-time time-varying linear systems are provided for ease of reference.

Consider the system

$$x(t+1) = A(t)x(t) + B(t)u(t),$$

$$y(t) = C(t)x(t),$$

where $t \in Z$ (the set of integers), $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$, and $A(\cdot), B(\cdot), C(\cdot)$ are matrix functions of suitable dimensions. The matrix $\Phi(t,\tau) = A(t-1)A(t-2)\cdots A(\tau)$, $t > \tau$, and $\Phi(t,\tau) = I$, $t = \tau$, is called the system transition matrix.

*Observability*

The state $z$ is unobservable over the time interval $[\tau, t]$, $\tau < t$, if to the free motion starting at $x(\tau) = z$ there corresponds $y(j) = 0$, $\forall j \in [\tau, t-1]$. The set of states that are unobservable over $[\tau, t]$ is a subspace $\bar{X}^{\omega}(\tau, t)$ called the unobservability subspace over $[\tau, t]$.

The state $z$ is unobservable at time $\tau$ if $z \in \bar{X}^{\omega}(\tau, t)$ $\forall t > \tau$. The set of states that are unobservable at $\tau$ is a subspace $\bar{X}^{\omega}(\tau)$ called the unobservability subspace at $\tau$.

The subspaces $X^{\omega}(\tau, t) = \bar{X}^{\omega}(\tau, t)^{\perp}$ and $X^{\omega}(\tau) = \bar{X}^{\omega}(\tau)^{\perp}$ are usually referred to as the observability subspaces over $[\tau, t]$ and at time $\tau$, respectively.

The system is said to be completely observable at time $\tau$ if $X^{\omega}(\tau) = R^n$. Finally, the system is said to be completely observable when $X^{\omega}(\tau) = R^n$ $\forall \tau$.

It is easy to prove (see e.g. [15]) that $\bar{X}^{\omega}(\tau, t) = N[W(\tau, t)]$, where,

$$W(\tau, t) = \sum_{j=\tau}^{t-1} \Phi(j,\tau)'C(j)'C(j)\Phi(j,\tau), \qquad \tau < t,$$

is the observability Gramian associated to the system.

*Reconstructibility*

The state $z$ is unreconstructible over the time interval $[\tau, t]$, $\tau < t$, if there exists a free motion ending in $x(t) = z$ that results in $y(j) = 0$ $\forall j \in [\tau, t-1]$. The set of states that are unreconstructible over $[\tau, t]$ is a subspace $\bar{X}^{\rho}(\tau, t)$ called the unreconstructibility subspace over $[\tau, t]$.

The state $z$ is unreconstructible at time $t$ if $z \in \bar{X}^{\rho}(\tau, t)$ $\forall \tau < t$. The set of states that are unreconstructible at $t$ is a subspace $\bar{X}^{\rho}(t)$ called the unreconstructibility subspace at $t$.

The subspaces $X^{\rho}(\tau, t) = \bar{X}^{\rho}(\tau, t)^{\perp}$ and $X^{\rho}(t) = \bar{X}^{\rho}(t)^{\perp}$ are usually referred to as the reconstructibility subspaces over $[\tau, t]$ and at time $t$, respectively.

The system is said to be completely reconstructible at time $t$ if $X^{\rho}(t) = R^n$. Finally, the system is said to be completely reconstructible when $X^{\rho}(t) = R^n$ $\forall t$.

In general, it is not possible to define a reconstructibility Gramian. The definition can be given for reversible systems only.

A comprehensive treatment of the structure theory of discrete-time linear systems with a detailed discussion of the notions of observability and reconstructibility (as well as the dual notions of reachability and controllability) is reported in [5].

*Detectability*

A precise definition of detectability can be found in [1]. In the particular case of periodic systems, this notion can be easily interpreted in terms of the Kalman canonical decomposition of the system. Precisely, the following characterization holds: The system is detectable if and only if its unreconstructible part is asymptotically stable.

For further details on different characterizations of the detectability of periodic systems see [9].

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